No. 3527, Contract No. DA 36-039 SC-87398, sec. 15.01, pp. 881–887; January, 1963.
[5] L. Young, "The application of branch-guide couplers to the suppression of spurious frequencies," *Proc. Fourth Nat'l Symp. on Radio Frequency Interference;* June 28–29, 1962.
[6] ——, "Synchronous branch guide directional couplers for low and high power applications," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-10, pp. 459–475; November, 1962.
[7] ——, "Waveguide 0-db and 3-db directional couplers as harmonic pads," *Microwave J.;* March, 1964.
[8] J. Collard, "The enthrakometer, an instrument for the measurement of power in rectangular wave guides," *Proc. IEE,* vol. 93, pt. III-A, pp. 1399–1402; 1946.
[9] L. J. Hinton, "Discussion on power measurement," *Proc. IEE,* vol. 109, pt. B, suppl. no. 23, pp. 757 and 759; September, 1962.
[10] J. A. Lane, "Transverse film bolometers for the measurement of power in rectangular waveguides," *Proc. IEE,* vol. 105, pt. B, pp. 77–80; January, 1958.
[11] I. Lemco and B. Rogal, "Resistive film milli-wattmeters for the frequency bands 8.2–12.4 Gc/s and 26.5–40 Gc/s," *Proc. IEE,* vol. 107, pt. B, pp. 427–430; September, 1960.
[12] J. A. Lane and D. M. Evans, "The design and performance of transverse-film bolometers in rectangular waveguides," *Proc. IEE,* vol. 108, pt. B, pp. 133–135; January, 1961.
[13] L. E. Norton, "Broad-band power-measuring methods at microwave frequencies," PROC. IRE, vol. 37, pp. 759–766; July, 1949.
[14] B. H. Billings, W. L. Hyde, and E. E. Barr, "An investigation of the properties of evaporated metal bolometers," *J. Opt. Soc. Am.,* vol. 37, pp. 123–132; March, 1947.
[15] H. J. Carlin and M. Sucher, "Accuracy of bolometric power measurements," PROC. IRE, vol. 40, pp. 1042–1048; September, 1952.
[16] M. V. Schneider, "Reflection and transmission of conductive films," PROC. IRE (*Correspondence*), vol. 49, pp. 1090–1091; June, 1961.
[17] T. P. Vogl, G. A. Shifrin, and B. J. Leon, "Generalized theory of metal-film bolometers," *J. Op. Soc. Am.,* vol. 52, pp. 957–964; September, 1962.
[18] J. Brown, "Reactive effects in transverse-film bolometers in rectangular waveguides," *Proc. IEE,* vol. 110, pt. B, pp. 77–78; January, 1963.
[19] "NBS finds error in a widely used test of UHF power," News-item, *Electronic Design,* April 13, 1963.

Five Layer Optical Maser Amplification

H. JACOBS, SENIOR MEMBER, IEEE, D. HOLMES, F. A. BRAND, SENIOR MEMBER, IEEE, AND L. HATKIN, MEMBER, IEEE

Summary—**The optical maser is treated in the manner of a Fabry-Perot resonator with an active medium. Five layers are considered: air, reflector, active medium (ruby), reflector, and air. General equations are derived using the method of boundary value problems in which it is assumed that incident coherent radiation falls normally on the surface. It is suggested that the presence of lossless one-quarter wavelength reflectors will enhance the amplification of the device in that less pumping may be required for a given length of ruby. The role of the reflectors in oscillation conditions is shown to be of importance. Methods are indicated for the calculation of amplitude and phase for an idealized amplifier.**

INTRODUCTION

CONSIDERABLE work has been reported on a three layer optical maser amplifier consisting of air, ruby and air,[1] treating the system as a transmission line, or boundary value problem in electromagnetic theory. Both approaches lead to the same equations for amplification and oscillation. Furthermore, these equations are equivalent to those developed by V. N. Smiley[2] for the same boundary conditions (system composed of air, ruby, air). In the latter work the analysis was based on a Fabry-Perot structure containing an active medium. In the following analysis, the calculations are carried further in that the multiple internal reflections in the reflectors are considered together with the internal reflections in the active medium. The optical maser amplifier is conceived as a five layer structure consisting of air, reflector, active material such as ruby, reflector, and air. General expressions are developed to represent transmitted gain in such a system. The special case is then examined in which the reflectors are lossless and are one-quarter wavelength in thickness. This system is then reduced to an equivalent three layer structure. Numerical calculations predict that for reflector materials with high dielectric constants and negligible loss, the length of the crystal, or the pumping power required for a given amplification, can be substantially reduced. One might have thought this possible since the quarter wavelength reflectors increase multiple internal reflections which in turn provide a superposition of waves resulting in greater power output. However, other factors enter the situation. A critical length is calculated for a given negative attenuation which produces maximum gain or oscillations. At lower values of length the gain in transmission increases with length. As the length increases beyond the critical value, the transmitted gain decreases.

Calculations are also made on the phase shift of the transmitted electric field compared with incident elec-

Manuscript received August 1, 1963; revised October 7, 1963.
Messrs. Jacobs, Brand, and Hatkin are with the U. S. Army Electronics Research and Development Laboratories, Fort Monmouth, N. J.
Mr. Holmes is with the Department of Electrical Engineering, Carnegie Institute of Technology, Pittsburgh, Pa.
[1] H. Jacobs, L. Hatkin, D. Holmes and F. A. Brand, "Laser amplifier design theory," Northeast Electronics Research and Engineering Meeting Record, Boston Section, Institute of Radio Engineers, Boston, Mass.; November, 1962.
[2] V. N. Smiley, "Air active interference filter as an optical maser amplifier," PROC IEEE, vol. 51, pp. 120–124; January 1963.

tric field. It is shown that the phase angle of the output wave at first lags that of the incident wave with increasing length, but beyond the critical length of oscillation or maximum gain the output actually leads the incident wave with increasing length. This is the same effect as if a wave in the active medium started on the output side and grew as it traveled in the reverse direction to the front surface. This phase effect is consistent with the fact that at very large lengths the transmitted wave approaches zero, while the reflected wave approaches an asymptotic limit dependent upon the impedances seen at the front surface.

In the following work an analysis is made indicating a method of designing an amplifier system consisting of multilayers. Consider an amplifier or oscillator with the following design. We have five layers as indicated in Fig. 1, each layer being infinite in the x, y plane and finite in the z direction. Region I refers to air, region II is a reflector, III a distributed negative conductivity medium such as an optical maser material, IV a reflector, and V air again. A general treatment of multiple layers is given by Weinstein.[3] However, for this particular system, one of the specific methods of calculating the ratio of transmitted electric field to incident electric fields together with phase shift will be described.

The equations for wave motion in each medium are given as follows, assuming a time variation of $e^{j\omega t}$, where w is assumed to be a constant angular frequency.

In medium I,

$$E_y = E_{\text{in}} e^{-j\beta_0 z} + E_R e^{j\beta_0 z} \tag{1a}$$

and

$$-H_x = \frac{j\beta_0}{j\omega\mu}\left(E_{\text{in}} e^{-j\beta_0 z} - E_R e^{j\beta_0 z}\right); \tag{1b}$$

in medium II,

$$E_y = E_1 e^{-\gamma z} + E_2 e^{\gamma z} \tag{1c}$$

and

$$-H_x = \frac{\gamma}{j\omega\mu}\left(E_1 e^{-\gamma z} - E_2 e^{\gamma z}\right); \tag{1d}$$

in medium III,

$$E_y = E_3 e^{-\Gamma z} + E_4 e^{\Gamma z} \tag{1e}$$

and

$$-H_x = \frac{\Gamma}{j\omega\mu}\left(E_3 e^{-\Gamma z} - E_4 e^{\Gamma z}\right); \tag{1f}$$

in medium IV,

$$E_y = E_5 e^{-\nu z} + E_6 e^{\nu z} \tag{1g}$$

[3] W. Weinstein, "The reflectivity and transmissivity of multiple thin coatings," *J. Opt. Soc. Amer.*, vol. 37, no. 7, pp. 576–781; July, 1947.

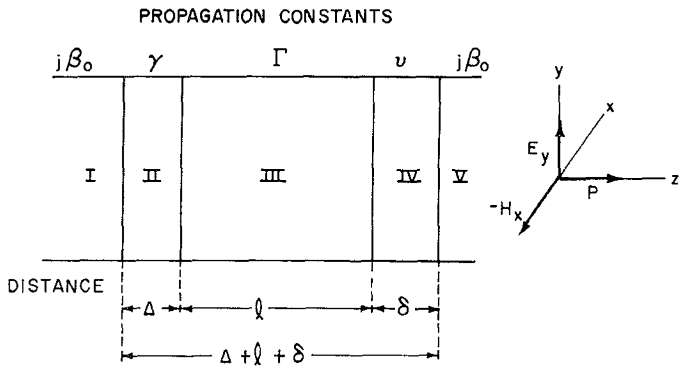

Fig. 1—Five layer structure for optical maser amplifier consisting of air, reflector medium, active material such as ruby, reflector and air. The incoming wave is assumed to be at normal incidence and single frequency. The dimensions are infinite in the x, y plane.

and

$$-H_x = \frac{\nu}{j\omega\mu}\left(E_5 e^{-\nu z} - E_6 e^{\nu z}\right); \tag{1h}$$

and in medium V,

$$E_y = E_0 e^{-j\beta_0} \tag{1i}$$

and

$$-H_x = \frac{j\beta_0}{j\omega\mu} E_0 e^{-j\beta_0}. \tag{1j}$$

Next we assume the following boundary conditions. At each interface the E fields are equal on both sides of the boundary and the H fields are equal. For instance, at $z=0$,

$$E_R - E_1 - E_2 + E_{\text{in}} = 0,$$

and

$$-j\beta_0 E_R - \gamma E_1 + \gamma E_2 - j\beta_0 E_{\text{in}} = 0; \tag{2a}$$

at $z=\Delta$,

$$E_1 e^{-\gamma\Delta} + E_2 e^{\gamma\Delta} - E_3 e^{-\Gamma\Delta} - E_4 e^{\Gamma\Delta} = 0$$

and

$$\gamma E_1 e^{-\gamma\Delta} - \gamma E_2 e^{\gamma\Delta} - \Gamma E_3 e^{-\Gamma\Delta} + \Gamma E_4 e^{\Gamma\Delta} = 0; \tag{2b}$$

at $z=\Delta+l$,

$$E_3 e^{-\Gamma(\Delta+l)} + E_4 e^{\Gamma(\Delta+l)} - E_5 e^{-\nu(\Delta+l)} - E_6 e^{\nu(\Delta+l)} = 0$$

and

$$\Gamma E_3 e^{-\Gamma(\Delta+l)} - \Gamma E_4 e^{\Gamma(\Delta+l)} - \nu E_5 e^{-\nu(\Delta+l)} + \nu E_6 e^{\nu(\Delta+l)} = 0; \tag{2c}$$

and at $z=\Delta+l+\delta$,

$$E_5 e^{-\nu(\Delta+l+\delta)} + E_6 e^{\nu(\Delta+l+\delta)} - E_0 e^{-j\beta_0(\Delta+l+\delta)} = 0$$

and

$$\nu E_5 e^{-\nu(\Delta+l+\delta)} - \nu E_6 e^{\nu(\Delta+l+\delta)} - j\beta_0 E_0 e^{-j\beta_0(\Delta+l+\delta)} = 0. \tag{2d}$$

Upon solving these equations simultaneously, we obtain the following equation for the ratio of electric field incident to the electric field appearing at the outer surface $(\Delta+l+\delta)$ in free space,

$$\begin{aligned}\frac{E_{in}}{E_o} = \cosh \Gamma l \Big[& \cosh \gamma\Delta \cosh \delta\nu \\ & + \frac{\nu^2 + (j\beta_0)^2}{2j\beta_0\nu} \cosh \gamma\Delta \sinh \nu\delta \\ & + \frac{\gamma^2 + (j\beta_0)^2}{2j\beta_0\gamma} \sinh \gamma\Delta \cosh \nu\delta \\ & + \frac{\nu^2 + \gamma^2}{2\nu\gamma} \sinh \gamma\Delta \sinh \nu\delta \Big] \\ & + \sinh \Gamma l \Big[\frac{1}{2}\left(\frac{\Gamma}{j\beta_0} + \frac{j\beta_0}{\Gamma}\right) \cosh \gamma\Delta \cosh \nu\delta \\ & + \frac{\Gamma^2 + \nu^2}{2\Gamma\nu} \cosh \gamma\Delta \sinh \nu\delta \\ & + \frac{\Gamma^2 + \gamma^2}{2\Gamma\gamma} \sinh \gamma\Delta \cosh \nu\delta \\ & + \frac{(j\beta_0\Gamma)^2 + (\gamma\nu)^2}{2j\beta_0\Gamma\gamma\nu} \sinh \gamma\Delta \sinh \nu\delta \Big], \end{aligned} \tag{3}$$

where $E_0 = E_0' e^{-j\beta_0(\Delta+l+\delta)}$.

If we assume the mirrors are of equal thickness, $\Delta=\delta$, and furthermore, of identical materials, $\nu=\gamma$, we obtain

$$\begin{aligned}\frac{E_{in}}{E_o} = \cosh \Gamma l \Big[& \cosh^2 \gamma\Delta + \frac{\gamma^2 + (j\beta_0)^2}{j\beta_0\gamma} \cosh \gamma\Delta \sinh \gamma\Delta \\ & + \sinh^2 \gamma\Delta \Big] + \sinh \Gamma l \Big[\frac{1}{2}\left(\frac{\Gamma}{j\beta_0} + \frac{j\beta^0}{\Gamma}\right) \cosh^2 \gamma\Delta \\ & + \frac{\Gamma^2 + \gamma^2}{\Gamma\gamma} \cosh \gamma\Delta \sinh \gamma\Delta \\ & + \frac{(j\beta_0\Gamma)^2 + \gamma^4}{2j\beta_0\Gamma\gamma^2} \sinh^2 \gamma\Delta \Big], \end{aligned} \tag{4}$$

where E_i is the incident electric field at $z=0$, $\Gamma=\alpha+j\beta$ is the propagation constant in medium III, which will be the optical maser material, $\gamma=\nu$ refers to the propagation constant of the reflectors, $\delta=\Delta$ refers to the thickness of the reflectors, β_0 is the phase constant in air in media I and V, and E_0 is the output field in medium V at the distance $z=\Delta+l+\delta$.

In (1)–(4), γ would be generally complex, *i.e.*, $\gamma=\alpha'+j\beta'$, the real part representing the loss. If the mirrors are assumed lossless, $\alpha'=0$ and only $j\beta'$ remains. It is to be noted further that in the optical maser material where $\Gamma=\alpha+j\beta$, α will be negative indicating a negative conductivity and growth of the electric field as the wave traverses the material.

Approximate Equations for Gain with Quarter Wavelength Reflectors

Next we consider certain simplifying assumptions which will facilitate the use of (4) where a specific numerical example is applied. Physically, we first assume that the mirrors are lossless and hence wave properties in this medium are only dependent upon length and $\beta'=\omega\sqrt{\mu\epsilon'}$, where ϵ' is the permittivity. Secondly, we shall assume the length $\Delta=\frac{1}{4}\lambda'$, where λ' is the wavelength in the material. In (4) we shall define $\gamma=\gamma'$ as the propagation constant in the medium of the reflector.

For lossless reflectors, *i.e.*, media II and IV,

$$\gamma'\Delta = j\beta'\Delta = j\frac{2\pi}{\lambda'}\cdot\frac{\lambda'}{4} = j\frac{\pi}{2}, \tag{5}$$

and

$$\cosh^2 j\frac{\pi}{2} = 0, \qquad \sinh^2 j\frac{\pi}{2} = -1. \tag{6}$$

Now substitute (5) and (6) into (4) obtaining,

$$\frac{E_o}{E_{in}} = \frac{-1}{\cosh \Gamma l + \frac{1}{2}\left[\frac{(j\beta_0\Gamma)^2 + \gamma^4}{j\beta_0\Gamma\gamma^2}\right] \sinh \Gamma l}. \tag{7}$$

Next we can make a comparison with the case of a ruby amplifier with no reflectors, *i.e.*, $\Delta=0$. This was previously described[4] as a three layer amplifier consisting of an air, ruby, air system. For the case of no added reflectors, the latter system was analyzed to give

$$\frac{E_o}{E_{in}} = \frac{1}{\cosh \Gamma l + \frac{1}{2}\left[\frac{\Gamma}{\gamma_0} + \frac{\gamma_0}{\Gamma}\right] \sinh \Gamma l} \tag{8}$$

where $\Gamma=\alpha+j\beta$ for the ruby, $\gamma_0=j\beta_0$ is air, l is the length of the ruby crystal, E_{in} is the electric field incident on the front surface and E_o is the transmitted output field in air at $z=l$.

Comparing (7) and (8) we see that the same general form occurs. The difference between the two equations lies in two factors. First, the negative sign indicates a 180° phase shift which is related to the addition of quarter wavelength lossless reflectors. Second, the coefficient of the sinh Γl term in the denominator of (7) appears different from that in (8). However, this can be rewritten as,

$$\begin{aligned}\frac{1}{2}\left[\frac{(j\beta_0\Gamma)^2 + \gamma^4}{2j\beta_0\Gamma\gamma^2}\right] &= \frac{1}{2}\left[\frac{j\beta_0\Gamma}{\gamma^2} + \frac{\gamma^2}{j\beta_0\Gamma}\right] \\ &= \frac{1}{2}\left[\frac{\gamma_0\Gamma}{\gamma^2} + \frac{\gamma^2}{\gamma_0\Gamma}\right], \end{aligned} \tag{9}$$

[4] H. Jacobs, D. A. Holmes, L. Hatkin, and F. A. Brand, "Maximum gain for forward- and backward-wave optical maser amplifiers," *J. Appl. Phys.* vol. 34, pp. 2617–2624, September 1963; H. Jacobs, R. A. Bowden and L. Hatkin, "On an active interference filter as an optical maser amplifier," Proc. IEEE, vol. 51, p. 933, June, 1963.

where $\gamma_0 = j\beta_0$ is the propagation constant in air, γ is the propagation constant in the reflectors, and Γ is the propagation constant in medium III, the active region.

As a check, it can be noted that when $\gamma = \gamma_0$, (7) reduces to (8). Thus we have formulated the case of the five layer problem in such a manner that it is similar in form to the three layer case and may be treated more simply. Eq. (7) may now be rewritten as

$$\frac{E_o}{E_{in}} = \frac{-1}{\cosh \Gamma l + \frac{1}{2}\left[\frac{\Gamma}{\gamma^2/\gamma_0} + \frac{\gamma^2/\gamma_0}{\Gamma}\right] \sinh \Gamma l}, \qquad (10)$$

where

$$\frac{\gamma^2}{\gamma_0} = j\frac{\beta'^2}{\beta_0}.$$

Now we shall utilize the fact that the form of (10) is the same as that used in the previous study of the three layer problem.[4] First consider

$$\frac{1}{2}\left(\frac{Z_{02}}{Z_{01}} + \frac{Z_{01}}{Z_{02}}\right) = \frac{1}{2}\left\{\frac{\frac{j\omega\mu}{\Gamma}}{\frac{j\omega\mu}{\gamma^2/\gamma_0}} + \frac{\frac{j\omega\mu}{\gamma^2/\gamma_0}}{\frac{j\omega\mu}{\Gamma}}\right\}, \qquad (11)$$

where

$$Z_{02} = \frac{j\omega\mu}{\Gamma}, \qquad Z_{01} = \frac{j\omega\mu}{\gamma^2/\gamma_0}.$$

Then

$$\frac{1}{2}\left(\frac{Z_{02}}{Z_{01}} + \frac{Z_{01}}{Z_{02}}\right) = \frac{1}{2}\left(\frac{\gamma^2/\gamma_0}{\Gamma} + \frac{\Gamma}{\gamma^2/\gamma_0}\right). \qquad (12)$$

We have reduced the five layer case to the three layer form assuming an intrinsic impedance in the ruby Z_{02} and an artificial impedance Z_{01} at the boundary. We shall refer to Z_{01} as Z in the remainder of this paper.

In addition, let $Z_{02} = X + jY$, so that

$$\frac{1}{2}\left(\frac{Z_{02}}{Z_{01}} + \frac{Z_{01}}{Z_{02}}\right) = \frac{1}{2}\left(\frac{X + jY}{Z} + \frac{Z}{X + jY}\right), \qquad (13)$$

and

$$Z_{02} = \frac{j\omega\mu}{\alpha + j\beta} = \frac{\beta\omega\mu}{\alpha^2 + \beta^2} + j\frac{\alpha\omega\mu}{\alpha^2 + \beta^2}. \qquad (14)$$

For $\alpha \ll \beta$,

$$X = \frac{\omega\mu}{\beta}, \qquad Y = \frac{\omega\mu\alpha}{\beta^2} = \frac{\alpha}{\beta}X, \qquad (15)$$

and

$$Z = \frac{j\omega\mu}{\gamma^2/\gamma_0} = \frac{\omega\mu}{\omega^2\mu\epsilon'/\omega\sqrt{\mu\epsilon_0}} = \frac{1}{\epsilon_R'}\sqrt{\frac{\mu}{\epsilon_0}}, \qquad (16)$$

where ϵ_R' is the dielectric constant of the lossless reflector and ϵ_0 is the permittivity of free space. Note that previously, in the three layer case, $Z = \sqrt{\mu/\epsilon_0}$. Further, in the five layer case,

$$\frac{X}{Z} = \frac{\frac{\omega\mu}{\beta}}{\frac{1}{\epsilon_R'}\sqrt{\frac{\mu}{\epsilon_0}}} = \frac{\epsilon_R'}{\sqrt{\epsilon_R}} \qquad (17)$$

$$\frac{Y}{Z} = \frac{\alpha}{\beta}\frac{X}{Z} = \frac{\alpha}{\beta}\frac{\epsilon_R'}{\sqrt{\epsilon_R}} \qquad (18)$$

where ϵ_R is the dielectric constant for ruby (or the active medium III) and ϵ_R' is the dielectric constant of the lossless reflector media (II and IV). Having determined X/Z and Y/Z, (17) and (18) can be used to find

$$\frac{1}{2}\left(\frac{Z_{02}}{Z_{01}} + \frac{Z_{01}}{Z_{02}}\right).$$

Eq. (13) can then be substituted into (10) and the magnitude $|E_o/E_{in}|^2$ can be found:

$$\left|\frac{E_o}{E_{in}}\right|^2 = \{[\cosh \alpha l \cos \beta l + K_1 \sinh \alpha l \cos \beta l - K_2 \cosh \alpha l \sin \beta l]^2 + [\sinh \alpha l \sin \beta l + K_1 \cosh \alpha l \sin \beta l + K_2 \sinh \alpha l \cos \beta l]^2\}^{-1} \qquad (19)$$

where

$$K_1 = \frac{X(X^2 + Y^2 + Z^2)}{2Z(X^2 + Y)^2}, \qquad K_2 = \frac{Y(X^2 + Y^2 - Z^2)}{2Z(X^2 + Y^2)}.$$

This procedure is similar to that already described in Jacobs, *et al.*[4] It can be shown that for $|\alpha| \ll \beta$, $K_1 = \frac{1}{2}(Z/X + X/Z)$, or

$$K_1 = \frac{1}{2}\frac{\epsilon_R'^2 + \epsilon_R}{\sqrt{\epsilon_R}\,\epsilon_R'}, \qquad (20)$$

and

$$K_2 = \frac{Y}{X}\frac{1}{2}\left(\frac{X}{Z} - \frac{Z}{X}\right) = \frac{\alpha}{\beta}\left[\frac{(\epsilon_R')^2 - \epsilon_R}{2\sqrt{\epsilon_R}\,\epsilon_R'}\right]. \qquad (21)$$

Conditions for Oscillation

As previously indicated, the criteria for maximum gain[5] or oscillation is given by

$$\coth \alpha l = -K_1, \qquad (22)$$

where α is the negative attenuation constant, l the length of the active medium III, and K_1 is a function of

[5] It turns out that for short lengths and large α values, a finite gain is obtained at the point of maximum gain for a given αl. However, as the length is increased, αl is constant (1.285 for ruby) for maximum gain, and the gain increases to infinity. This is then considered as an oscillation.

dielectric constants (20), with the assumption that $|\alpha| \ll \beta$. Rewriting (22),

$$\tanh \alpha l_5 = \frac{-2\sqrt{\epsilon_R}\,\epsilon_R'}{(\epsilon_R')^2 + \epsilon_R}, \tag{23}$$

where the subscript 5 on the left side of (23) denotes that we are referring to the five layer problem, that is, the ruby with quarter wavelength lossless reflectors. By comparison, for the three layer case with no reflecting mirrors,

$$\tanh \alpha l_3 = \frac{-2\sqrt{\epsilon_R}}{\epsilon_R + 1} \tag{24}$$

is the criterion for oscillation. It is of interest to note that (24) is in agreement with the work of Maiman and of Smiley.[2]

In (23), if we let $\epsilon_R' = \eta\epsilon_R$, so that η is the ratio of the dielectric constant of the mirror to that of the ruby, we find

$$\rho \equiv \frac{\tanh \alpha l_5}{\tanh \alpha l_3} = \frac{\eta(\epsilon_R + 1)}{\eta^2\epsilon_R + 1}. \tag{25}$$

This indicates that for high η values ρ can become considerably smaller than one, and $|\alpha l_5| < |\alpha l_3|$ for oscillations. This has the desirable feature that when reflectors are used, either the pumping ($|\alpha|$) can be smaller or the length smaller. Similarly, high amplification can result at lower αl values for reflectors with large values of η.

As an example of the use of (25), Fig. 2 illustrates how ρ changes as a function of η. For practical purposes, materials with an η value near 2 are readily available.

More specifically, let us consider the theoretical oscillation point for the case of zinc sulfide films one-quarter wavelength thickness on the end surfaces of a ruby rod. Here $\sqrt{\epsilon_R} = 1.765$ for ruby, and $\sqrt{\epsilon_R'} = 2.368$ for zinc sulfide. Hence, for the three layer case, by (24)

$$\begin{aligned} \tanh \alpha l_3 &= -0.8578 \\ \alpha l_3 &= -1.285 \end{aligned} \tag{26}$$

for oscillations and,

$$\begin{aligned} \tanh \alpha l_5 &= -0.5728 \\ \alpha l_5 &= -0.6517 \end{aligned} \tag{27}$$

or

$$\frac{\alpha l_5}{\alpha l_3} = 0.535. \tag{28}$$

This indicates that the oscillation point for the zinc sulfide coated crystal would require about half the value of length of ruby crystal or half the negative attenuation (measure of pumping power) of that needed by the three layer system.

Recognizing that the above approach to the calculation of the oscillation condition can be shown to be equivalent to the condition postulated by Smiley,

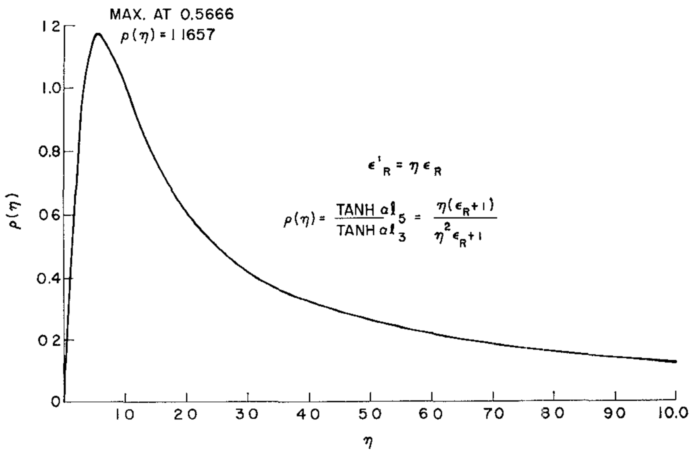

Fig. 2—The ratio

$$\rho = \frac{\tanh \alpha l_5}{\tanh \alpha l_3}$$

as a function of η. The term $\tanh \alpha l_5$ indicates the value required for maximum amplification or oscillation of a five layer system. The $\tanh \alpha l_3$ term is that required for a three layer system. In this case we refer to a five layer system with lossless quarter wavelength reflectors the dielectric constant of which is $\epsilon_R' = \eta\epsilon_R$, where ϵ_R = the dielectric constant of the active region (medium III).

further concepts can be developed. In Smiley's analysis, the criteria for oscillation was

$$R_P e^{-\Gamma' d} - 1 = 0 \tag{29}$$

where $\Gamma' = 2\alpha$, and R_P was the reflection coefficient for power when considering an electromagnetic wave in the ruby striking the reflector and being partially reflected back into the ruby again. Now $\sqrt{R_P}$ can be defined as

$$\sqrt{R_P} = \left[\frac{Z_{01} - Z_{02}}{Z_{01} + Z_{02}}\right], \tag{30}$$

where Z_{02} is the impedance of a wave in ruby and Z_{01} (or Z) is the impedance that the wave sees at the boundary entering the reflecting coating.

By (30) and the analog set up in (11) through (17),

$$\sqrt{R_P} = \left[\frac{\dfrac{j\omega\mu}{\gamma^2/\gamma_0} - \dfrac{j\omega\mu}{\Gamma}}{\dfrac{j\omega\mu}{\gamma^2/\gamma_0} + \dfrac{j\omega\mu}{\Gamma}}\right] = \left[\frac{\Gamma - \gamma^2/\gamma_0}{\Gamma + \gamma^2/\gamma_0}\right], \tag{31}$$

where $\Gamma = \alpha + j\beta \to j\beta$ in ruby as $\alpha \to 0$, $\gamma = j\beta'$ in the reflector and $\gamma_0 = j\beta_0$ in air. For $\alpha \to 0$ in the ruby and $\alpha' = 0$ in the reflectors,

$$\sqrt{R_P} = \frac{j\beta - j\dfrac{(\beta')^2}{\beta_0}}{j\beta + j\dfrac{(\beta')^2}{\beta_0}}, \tag{32}$$

where $\beta_0 = \omega\sqrt{\mu\epsilon_0}$ in air, $\beta' = \omega\sqrt{\mu\epsilon_R'\epsilon_0}$ in the reflector, $\beta = \omega\sqrt{\mu\epsilon_R\epsilon_0}$ in ruby, and

$$\sqrt{R_P} = \frac{\sqrt{\epsilon_R} - \epsilon_R'}{\sqrt{\epsilon_R} + \epsilon_R'}. \tag{33}$$

Here (33) is of the same form as if the ruby were imbedded in a dielectric medium $\epsilon_R'^2$ of infinite extent. We conclude that placing quarter wavelength coatings on the optical maser crystal has the same effect on internal reflections as if the dielectric $\epsilon_R'^2$ were infinite in length when considering oscillations in the active material, medium III.

The Role of Quarter Wavelength Reflectors on Amplification

For $|\alpha| \ll \beta$, in the ruby crystal, it is possible to postulate a conductivity such that

$$\alpha \cong \frac{\sigma}{2} Z_{02}, \quad \text{or} \quad \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon_0\epsilon_R}}, \tag{34}$$

or for larger α values, one can use the more exact equations which relate α and σ,

$$\alpha = -\frac{1}{\sqrt{2}}\left\{-\omega^2\mu\epsilon + \left[(\omega^2\mu\epsilon)^2 + (\sigma\omega\mu)^2\right]^{1/2}\right\}^{1/2} \tag{35}$$

$$\beta = +\frac{1}{\sqrt{2}}\left\{+\omega^2\mu\epsilon + \left[(\omega^2\mu\epsilon)^2 + (\sigma\omega\mu)^2\right]^{1/2}\right\}^{1/2}. \tag{36}$$

In the previous work, using values of $\sigma = -3000$ in (34), the transmission coefficient E_o/E_{in} was calculated. This was checked by using the more exact form of (35) and (36) and substituting in (8). The magnitude of E_o/E_{in} as well as its phase angle could thus be determined. Computer data are shown in Figs. 3 and 4. In Figs. 5 and 6, we see the result for the same value for $\sigma = -3000$, but with quarter wavelength reflectors added with $\sqrt{\epsilon_R'} = 2.368$. The oscillation or maximum gain point has moved to the left as predicted for the five layer case. This also results in increased gain for the amplifier, in which case operation usually occurs to the left of the oscillation point, *i.e.*, $|\alpha l| < |\alpha l_{max}|$.

In dealing with experimental optical masers, the values of α are always lower and the length l longer than indicated in the computer data.

In almost all previous reports on amplifiers where longer lengths were used in order to match conditions for specific experiments, information could be obtained about the specific length chosen, such as gain, bandwidth, etc. But in concentrating on a single region, much information was lost which could otherwise be obtained by compressing the length and increasing $|\alpha|$. The data in Figs. 3, 4, 5, and 6 show over-all trends not previously recognized. For instance, the maximum points are indicated and the decline in transmission is shown as l is increased. Furthermore, the relationship of phase angle being first negative in slope and later positive in slope with respect to increasing length of ruby is clearly shown.

Although the computer data shows only general trends, the exact equations developed can also be used to study the situation when $|\alpha|$ is smaller as in the case of specific experiments. We shall next show how α can be more accurately calculated using typical data obtained in experiments. As an example, let us consider the following: assume a ruby crystal of 0.025 m length that has given power gain of 2 in light intensity at a single frequency ω of light passing normally into the front surface of an amplifier using no reflectors. Using a former theory,[6] which did not take into account any multiple reflections,

$$I = I_0 e^{-2\alpha l} \tag{37}$$

where I is the intensity; if $I/I_0 = 2$, $l = 0.025$ meter, then α would be calculated as -13.3. For the three layer case,[4]

$$G_P = \frac{1}{\left[(\cosh \alpha l + K_1 \sinh \alpha l)^2 + (K_2 \sinh \alpha l)^2\right]}, \tag{38}$$

if $G_P = 2$, $l = 0.025$ meter, α would be equal to -11.26, a smaller value due to multiple internal reflections. In this case,

$$K_2 \to 0, \qquad K_1 \to \frac{2\sqrt{\epsilon_R}}{\epsilon_R + 1}.$$

We conclude that the determination of $|\alpha|$ is less when considering multiple reflections than would be estimated by assuming single pass amplification.

For the five layer case with zinc sulfide quarter wavelength reflectors added, for a gain of 2, α would be approximately -7.028. Using (38) for this calculation,

$$K_1 = \frac{2\sqrt{\epsilon_R}\,\epsilon_R'}{(\epsilon_R')^2 - \epsilon_R}$$

taking into account the presence of the reflectors, and $K_2 = 0$ for $|\alpha| \ll \beta$. This indicates that if the zinc sulfide reflectors of quarter wavelengths are added, the pumping required (which is related to $|\alpha|$) could be considerably reduced.

It was first thought that silver mirrors partially transmitting could also be added to increase amplification. It was calculated, however, that due to the losses in silver, amplification would not be enhanced. In Figs. 7 and 8, we see that due to the presence of silver, the gain has actually decreased for a σ value of -3000 comparable with the other figures. It should be pointed out that the case of silver mirror reflectors has been computed using (4) and assuming the bulk conductivity of silver in the complex propagation constants even for small values of thickness. Many other calculations with silver mirrors have been tried as well as experiments. In all cases, except where the silver coated rubies oscillate, the amplification was decreased due to the prescence of silver when compared with uncoated ruby.

[6] P. P. Kisliuk and W. S. Boyle, "The pulsed ruby maser as a light amplifier," Proc. IRE, vol. 49, pp. 1635–1639; November, 1961.

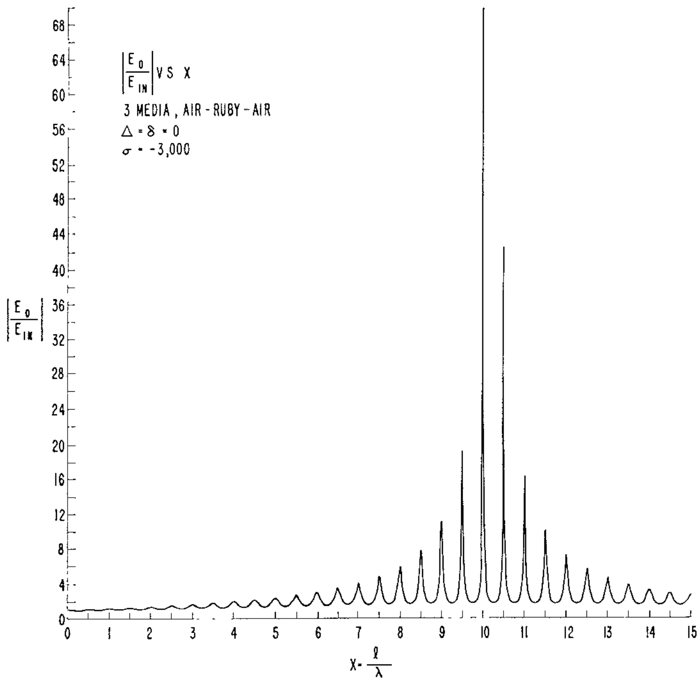

Fig. 3—The magnitude of E_o/E_{in} (the ratio of output field to incident field) for a three layer amplifier consisting of air, ruby, air, with the incoming wave at normal incidence. The active region is assumed to be ruby, $\omega = 2.714 \times 10^{15}$ and $\sigma = -3000$. Units of length are normalized where l is the length of active medium and λ the wavelength in the same material. As $|\alpha|$ decreases, the length for maximum gain increases as well as the magnitude of gain.

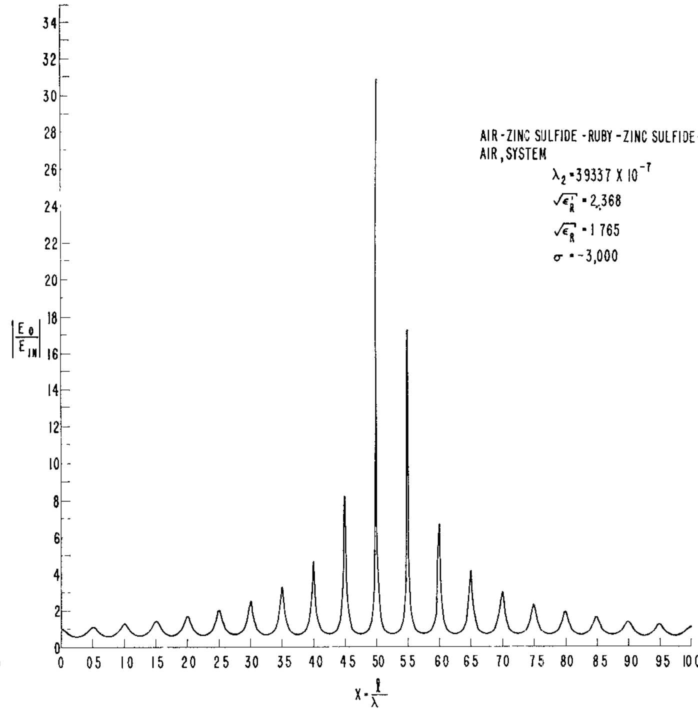

Fig. 5—The magnitude of E_o/E_{in} for a five layer system consisting of air reflector, ruby, reflector and air, as a function of length of the ruby crystal. The reflectors are assumed to be on quarter wavelength in thickness and made of lossless zinc sulfide for which $\sqrt{\epsilon_{R'}} = 2.368$. For comparison, the σ in ruby is maintained at -3000 as in Figs. 3 and 4, and the frequency is assumed the same. E_{in} is the incident electric field in air and E_o is the output field in air adjacent to the far side of the second reflector. Units of length are normalized where l is the length of the active material and λ is the wavelength in the same medium.

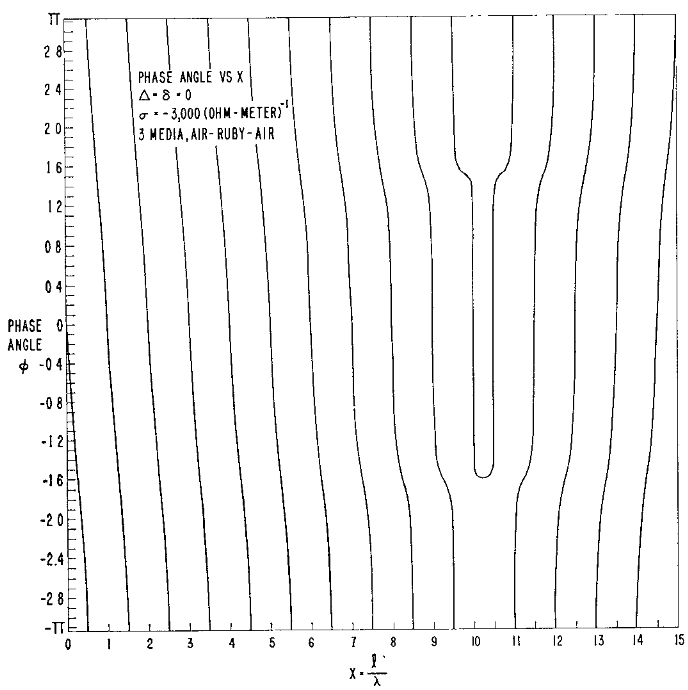

Fig. 4—The phase angle of E_o/E_{in} for a three layer system consisting of air, ruby and air. Here $\sigma = -3000$, $\omega = 2.714 \times 10^{15}$ as in Fig. 3.

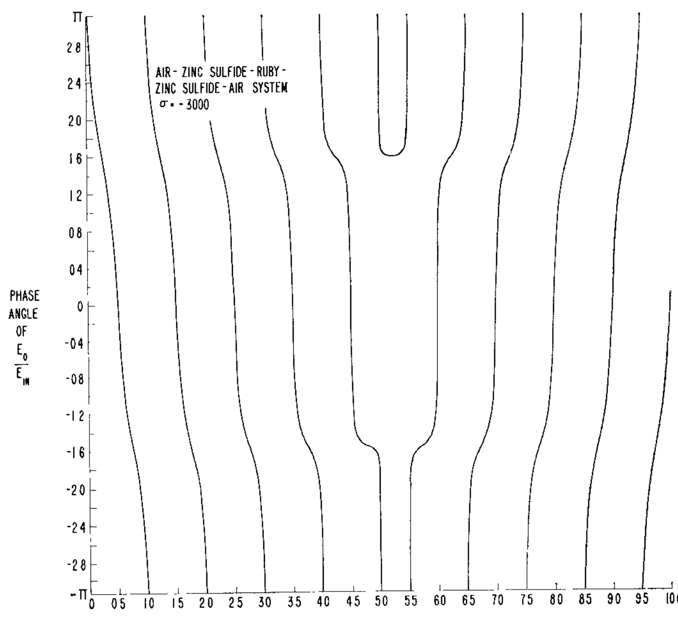

Fig. 6—The phase of E_o/E_{in} as a function of length in the five layer system shown in Fig. 5.

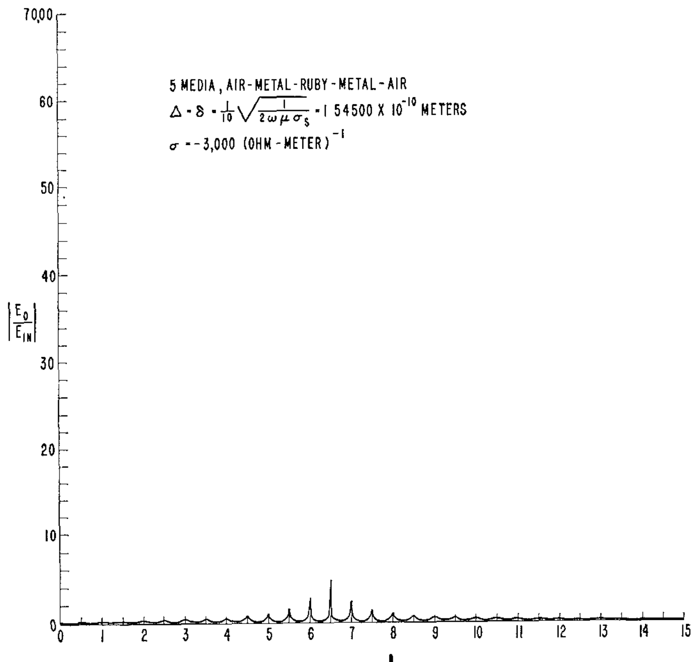

Fig. 7—The magnitude of E_o/E_{in} vs length for a five layer system consisting of air, silver film, ruby, silver film and air, as a function of length of the ruby crystal. To compute the propagation constant in silver, the bulk conductivity of silver was used. The value of $\sigma = -3000$ was again tried for ruby for comparison purposes and the thickness of the silver films assumed to be 1/10 of an electric field skin depth. Results indicate a considerable loss for the amplifier over all film thickness values computed due to losses in the metallic film.

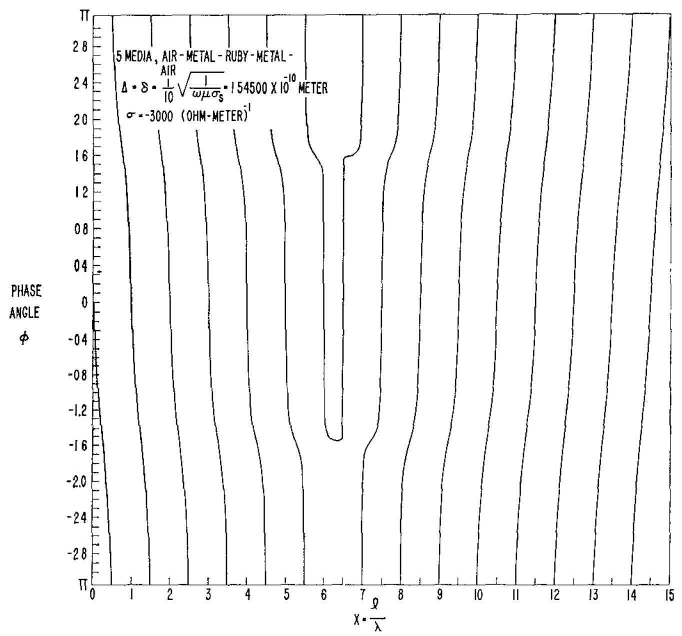

Fig. 8—The phase angle of E_o/E_{in} for the same case described by Fig. 7.

Conclusions

Calculations have been carried out which indicate a possible approach to the design of laser amplifiers using reflective coatings. It is suggested that odd multiple quarter wavelength reflectors will enhance the amplification characteristics of optical masers, providing the dielectric constant of the reflector is higher than that of the active medium. Conditions for oscillations are derived which are in agreement with previous work. Idealized conditions are used, such as single frequency, planar boundaries and homogeneous media. However, by using these conditions as a starting point to calculate gain and phase and to make comparisons of different structures, much can still be learned.

The fact that the gain should decline for long laser rods has come out of the calculations here in a manner consistent with previous reports.[4] Furthermore, this is implied in the equations given by Smiley although he did not specifically discuss the matter. The exact physical mechanism of this effect is not yet clarified. However, the authors are currently studying this effect from an impedance matching point of view. It is expected that the asymptotic value of reflection coefficient (reflected gain) can be deduced by the impedance that an incoming wave sees at the front surface of a long line with distributed negative conductance and that the transmitted wave can also be described by impedance matching concepts. Work is continuing on this aspect at the present time.

Appendix

Electromagnetic Constants for Ruby Optical Maser (MKS Units)

$c = 2.998 \times 10^8$ $\qquad \omega\epsilon = 7.481 \times 10^4$

$\lambda_{air} = 0.6943 \times 10^{-6}$ $\qquad \beta_{ruby} = \frac{2\pi}{\lambda_{ruby}} = 1.597 \times 10^7$

$\lambda_{ruby} = 0.3934 \times 10^{-6}$ $\qquad Z_{01}$ in air $= 376.7$

$f = 4.319 \times 10^{14}$ $\qquad \mu = 1.257 \times 10^{-6}$

$\omega = 2.714 \times 10^{15}$ $\qquad \frac{\pi}{\omega\epsilon\lambda_{ruby}} = 106.75$

$\omega\mu = 3.416 \times 10^9$ $\qquad \alpha = 106.75\sigma$ for ruby

$\epsilon_0 = 8.85 \times 10^{-12}$

$\sqrt{\epsilon_R} = 1.765$ for ruby $\qquad \left|\frac{\beta}{\alpha}\right| = \frac{1.5 \times 10^5}{|\sigma|}$

$\epsilon_R = 3.115$ $\qquad \beta_0 = \frac{2\pi}{\lambda_{air}} = 9.06 \times 10^5$

$\epsilon = \epsilon_R\epsilon_0$ $\qquad \sigma =$ effective conductivity

$= 2.757 \times 10^{11}$

Acknowledgment

The authors wish to thank R. Johnson and R. A. Bowden for their help on the computer calculations, and V. Kublin and Dr. E. A. Gerber for their discussions during the course of this work.